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Closed Form Solution for the Sonic Boom in a Polytropic Atmosphere

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The sonic boom problem for typical aircraft maneuvers in a polytropic atmosphere is solved analytically using the analytic method of characteristics. The linearized wave propagation, which serves as initial solution to the method of characteristics, is solved first. The velocity perturbations are multiplied by the factor $(c_{op}p_{op}/c_o p_o)^{1/2}$. With c_o as local sound velocity and p_o as local static pressure, the product $c_o p_o$ is, per unit time, the work done by the static pressure at local altitude, and $c_{op}p_{op}$ is the corresponding quantity at flight altitude. The characteristic method is modified to encompass the case of an oncoming stream with variable sound velocity.

Nomenclature

A = coefficient of sound-velocity statification
B = acceleration coefficient
b = acceleration
c = speed of sound
F = Whitham's F function
G = distance function
g = gravity acceleration
K_r = curvature of the envelope
$K_{1,2}$ = constants of integration
l = body length
M = Mach number related to c_{op}
m = gradient of the speed of sound
p = pressure
R_K = radius of curvature of the flight path
R = instantaneous radius of the wave front
s = ray
t = time dimensionless by l and c_{op}
u = velocity vector
x, y, z = rectangular coordinates dimensionless by l
x = axial coordinate
z = vertical coordinate
α = coefficient of gravity stratification
γ = ratio of specific heats
η = coefficient of polytropic stratification

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θ = lateral angle
μ, ν = bicharacteristics
ξ = source position
ρ = density
τ = time of the source
Φ = angle between flight path and horizontal plane
ϕ = retarded potential

Superscripts

$-$ = distorted quantities
$\sim$ = reduced quantities

Subscripts

o = local quantity of the polytropic atmosphere
1, 2 = first, second-order quantity
N = component in the direction of the wave-front normal
p = quantity at flight altitude
r = component in radial direction
t = component in tangential direction
x, y, z = vector components in x, y, z direction

Introduction

"THERE is a need for analytical methods that are capable of accurately predicting the near-flow region about arbitrary aircraft configurations. Available theories are either too cumbersome or limited in range of application." This is a statement of a paper presented at the Second Conference on Sonic Boom.[1] In the present paper an analytical perturbation method is outlined, which describes both the near-flow region and the far field in a polytropic atmosphere. This method predicts the near-flow region about a three-dimensional aircraft configuration with the same accuracy as those methods, which are valid only in a homogeneous atmosphere. The sonic boom propagation in a polytropic atmosphere is reduced to a propagation in an isothermal one, which has been solved by the author in a previous paper.[2]

General Description

In the analytical method of characteristics, characteristic manifolds are used as independent variables (see Refs. 3–7). Therefore, by straining the coordinates, regions of dependence and influence are obtained correctly, and in contrast to van Dyke's method,[8] the first-order solution is valid throughout the whole flow field. The linearized solution of the characteristic space serves as initial solution to the analytic method of characteristics. This solution, which is also called "acoustic solution," gives the correct first-order behavior of the flow on the characteristics.

In the next step, the correct first-order position of the characteristics will be obtained by means of the perturbed coordinates. Then shock waves can be constructed using Pfriem's formula. This procedure is similar to Whitham's for homogeneous atmosphere,[9] but because of the analytical method of characteristics (see Refs. 4–5), higher orders can be calculated,[10, 11] as well as interference phenomena between different families of characteristics[12, 13] and transonic problems.[14] Furthermore, the three-dimensional configuration of an aircraft can be taken into account, in particular in those cases where the equivalent body of revolution fails, as was shown by Oswatitsch and Sun for the sonic boom resulting from lift under a delta wing.[21, 22]

The present paper is restricted to the influence of a polytropic atmosphere. Therefore, only bodies of revolution are considered, though the method presented could be extended, for example, to the previously-mentioned delta wing. The calculations concerning the sonic boom are restricted to shock waves and their immediate neighborhood. Some results of this method were presented by the author as contributed remark at the Third Conference on Sonic Boom.[15] Here the method itself is presented.

Analytic Method of Characteristics

In the analytical method of characteristics of Oswatitsch,[5] bicharacteristics are used as independent variables. The physical coordinates (including the time)

$$x = x_0 + x_1 + x_2 + \cdots; \quad y = y_0 + y_1 + y_2 + \cdots; \quad z = z_0 + z_1 + z_2 + \cdots; \quad t = t_0 + t_1 + t_2 + \cdots; \tag{1}$$

as well as the velocity and the thermodynamic variables

$$p = p_0 + p_1 + p_2 + \cdots; \quad \rho = \rho_0 + \rho_1 + \rho_2 + \cdots; \quad c = c_0 + c_1 + c_2 + \cdots; \quad u_x = u_{1x} + u_{2x} + \cdots; \quad u_y = u_{1y} + u_{2y} \ldots; \quad u_z = u_{1z} + u_{2z} \ldots; \tag{2}$$

are expressed by power-series expansion in a small parameter, which is the thickness ratio in the case presented. There are no *zero*th-order velocities, since a wind-coordinate system is used.

The characteristic space indicated by the coordinates x_o, y_o, z_o and t_o coincides with the physical space indicated by x, y, z and t, if the perturbation coordinates x_i, y_i, z_i and t_i vanish. These perturbation coordinates follow from the relations for the characteristic directions and determine the position of the characteristics in the physical space.[5] The head wave is located very close to the characteristic front which separates the perturbed region from the undisturbed field (see Fig. 1).

The characteristic front—in the following, denoted as "acoustic perturbation front"—is shown for the vertical x_o, z_o plane above and below the horizontal flight path (see Fig. 1). It is calculated by means of geometric acoustics, and it is the envelope of Mach conoids with their tips on the flight path. The tips of the conoids are indicated by the points t_p, x_p with the static pressure p_{op} at the origin and the corresponding Mach number M. Though the method presented can be extended to atmospheric variations, a polytropic atmosphere with a linear sound velocity gradient

$$dc_o/dz_o = m \tag{3}$$

is used, since the corresponding geometric acoustics have been investigated extensively by Lansing.[16] According to the mathematical theory for envelopes in a parametric representation, the acoustic perturbation front is given by the equation of the Mach conoid

$$(x_o - x_p)^2 + \left(z_o - \frac{1}{A}\{\cosh|A(t_o - t_p)| - 1\}\right)^2 = \frac{1}{A^2}\sinh^2|A(t_o - t_p)| \tag{4}$$

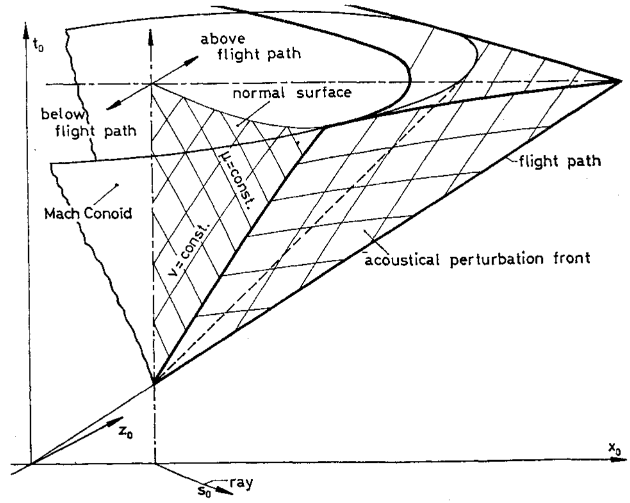

Fig. 1 The acoustical perturbation front is the envelope of Mach or Monge conoids. It is shown at different times t_o for the vertical x_o, z_o-plane, which contains the flight path.

Differentiation of Eq. (4) with respect to the parameter t_p yields

$$(x_o - x_{op})M + \left(- z_o + \frac{1}{A}\{\cosh|A(t_o - t_{op})| - 1\}\right) \times$$

$$\sinh|A(t_o - t_{op})| = \frac{1}{A}\sinh|A(t_o - t_{op})|\cosh|A(t_o - t_{op})| \quad (5)$$

where

$$x_p = t_p + B_t(t_p^2/2) \qquad M = 1 + B_t t_p \quad (6)$$

and

$$A = ml/c_{op} \quad (7)$$

$$B_t = bl/c_{op}^2 \quad (8)$$

Every Mach conoid has in common a tangent with the envelope (see Fig. 1), and is used as a parameter for the shock-wave propagation along this tangent. The tangent of the Mach conoid with the envelope lies in the bicharacteristic surface $\nu = 0$ and its projection on the $x_o - z_o$ plane is also denoted as ray s. The gradient of the envelope in sections $t_o = \text{const}$ is given from geometrical acoustics[16]

$$\frac{dz_o}{dx_o} = \left[\left(M\frac{c_{op}}{c_o}\right)^2 - 1\right]^{-1/2} \quad (9)$$

where $M\, c_{op}/c_o$ is the local Mach number. The curvature of the envelope in sections $t_o = \text{const}$ is given by

$$K_r = \left\{A\frac{z_0(M^2-1)^{1/2}}{M(x_o - x_p)} + \frac{B_t}{M}\left[\left(M\frac{c_{op}}{c_o}\right)^2 - 1\right]^{1/2}\right\}$$

$$\left\{\left[\left(M\frac{c_{op}}{c_o}\right)^2 - 1\right](M^2-1)^{1/2} - B_t(x_o - x_p) \times \left[\left(M\frac{c_{op}}{c_o}\right)^2 - 1\right]^{1/2}\right\}^{-1} \quad (10)$$

The plane perpendicular to the acoustic perturbation front (Fig. 1) is indicated by the ray and the axis of the Mach conoid. The bicharacteristics $\mu = \text{const}$ and $\nu = \text{const}$ are generated by intersections of bicharacteristic surfaces with the perpendicular plane. Thus, the problem depends only on the parameter t_p and the two variables μ and ν, since shocks are calculated in these perpendicular planes. The bicharacteristics $\mu = \text{const}$ are chosen so that along the $(t_o - t_p)$ axis they are perpendicular to the bicharacteristics $\nu = \text{const}$ (see Fig. 2a).

Let the distance measured along the ray from the origin be s, and introduce the abbreviations

$$\bar{s} = \int_o^s \frac{c_{op}}{c_o(s)}\,ds \quad (11)$$

and

$$\bar{t}_o = t_o - t_p$$

$\bar{s}$ represents a distorted ray, so that the bicharacteristics are perpendicular to each other over the $\bar{t}_o$, $\bar{s}$ plane (see Fig. 2b). The bicharacteristics are

$$2\mu = \bar{t}_o + \bar{s} \qquad 2\nu = \bar{t}_o - \bar{s} \quad (12)$$

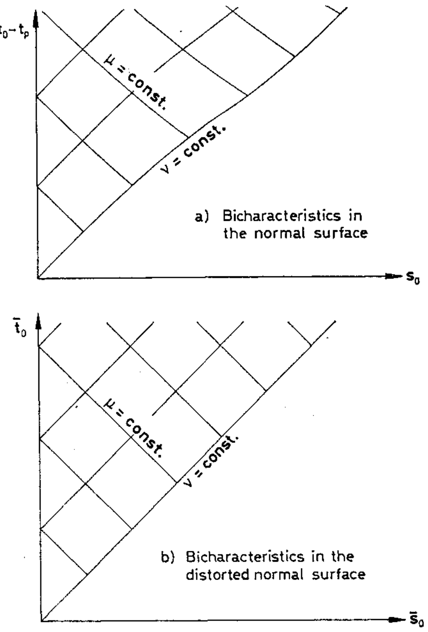

Fig. 2 The distance measured along the ray s_o (Fig. 2a) is distorted by means of $\bar{s}_o$ [Eq.(11)] such, that the bicharacteristics μ and ν are perpendicular to each other (Fig. 2b).

Now the zeroth-order coordinates can be calculated by means of geometrical acoustics[16]

$$\mu + \nu = \bar{t}_o$$

$$\mu - \nu = \bar{s} = (1/A)\,\text{arsinh}|AR|$$

R is defined by Eq. (26) (13)

$$x_o - x_{op} = \frac{1}{M}(c_o/c_{op})\frac{1}{A}\sinh|A\bar{t}_o|$$

$$z_o = (c_o/Ac_{op})[1 - (c_{op}/c_o)]$$

with

$$c_{op}/c_o = \cosh|A\bar{t}_o| - (1/M)(M^2-1)^{1/2}\sinh|A\bar{t}_o|$$

Using the wind-coordinate system, one can calculate the first-order coordinate perturbations after Oswatitsch[5]

$$s_1 - t_1 = \int_{\mu 0}^{\mu}\{u_{1N}(\bar{\mu},\nu) + c_1(\bar{\mu},\nu)\}d\bar{\mu} + K_1(\nu)$$

$$s_1 + t_1 = \int_{\nu 0}^{\nu}\{u_{1N}(\mu,\bar{\nu}) - c_1(\mu,\bar{\nu})\}d\bar{\nu} + K_2(\mu) \quad (14)$$

u_{1N} is the first-order perturbation velocity in the direction of the ray, which in a still atmosphere is always in the direction of the acoustic perturbation front normal. u_{1N} follows from the acoustic wave propagation in a polytropic atmosphere.

Analytic Solution of the Wave Propagation

The system of differential equations describing the wave propagation in a polytropic atmosphere can be linearized with respect to small perturbations. One then obtains the following differential equations for the conservation of mass, momentum and energy

$$\frac{1}{\rho_o}\left(c_{op}\frac{\partial\rho_1}{\partial t_o} + u_{1z}\frac{\partial\rho_o}{\partial z_o}\right) + \frac{\partial u_{1x}}{\partial_{xo}} + \frac{\partial u_{1y}}{\partial_{yo}} + \frac{\partial u_{1z}}{\partial_{zo}} = 0 \quad (15)$$

$$c_{op}(\partial u_{1x}/\partial t_o) = -(1/\rho_o)(\partial p_1/\partial x_o)$$

$$c_{op}(\partial u_{1y}/\partial t_o) = -(1/\rho_o)(\partial p_1/\partial y_o) \quad (16)$$

$$c_{op}(\partial u_{1z}/\partial t_o) = -[(1/\rho_o)(\partial p_1/\partial z_o)] - gl(\rho_1/\rho_o)$$

$$\frac{1}{\gamma p_o}\left(c_{op}\frac{\partial p_1}{\partial t_o} + u_{1z}\frac{\partial p_o}{\partial z_o}\right) = \frac{1}{\rho_o}\left(\frac{\partial\rho_1}{\partial t_o} + u_{1z}\frac{\partial\rho_o}{\partial z_o}\right) \quad (17)$$

Fundamental equation of static meterology:

$$(1/p_o)(dp_o/dz_o) = -gl \tag{18}$$

The gravity acceleration g enters the equations through the constant

$$\alpha = g\gamma l/c_{op}^2 \tag{19}$$

which is of the order of the body length divided by the virtual altitude of the homogeneous atmosphere.

Using a transformation similar to that one used in Refs. 2, 17, 18′

$$u_{1x,y,z}/c_o = (c_{op}p_{op}/c_o p_o)^{1/2}\, \tilde{u}_{1x,y,z}/c_{op} \tag{20}$$

and neglecting terms of the order α^2, the preceding conservation equations can be rewritten to give:

Continuity equation

$$\frac{1}{\gamma p_o}\frac{\partial p_1}{\partial t_o} = \left(\frac{c_{op}p_{op}}{c_o p_o}\right)^{1/2}\left(\frac{\partial \tilde{d}}{\partial t_o} - \left[\frac{gl}{c_o c_{op}}\frac{2-\gamma}{2} + \frac{1}{2c_{op}}\frac{dc_o}{dz_o}\right]\frac{\tilde{u}_{1z}}{c_{op}}\right) \tag{21}$$

where

$$\frac{c_{op}}{c_o}\frac{\partial \tilde{d}}{\partial t_o} = -\frac{1}{c_{op}}\left(\frac{\partial \tilde{u}_{1x}}{\partial x_o} + \frac{\partial \tilde{u}_{1y}}{\partial y_o} + \frac{\partial \tilde{u}_{1z}}{\partial z_o}\right) \tag{22}$$

Momentum equations

$$\begin{aligned}
\frac{1}{c_o}\frac{\partial^2 \tilde{u}_{1x}}{\partial t_o^2} &= -\frac{\partial^2 \tilde{d}}{\partial x_o \partial t_o} + \eta\frac{\partial}{\partial x_o}\left(\frac{\tilde{u}_{1z}}{c_{op}}\right)\\
\frac{1}{c_o}\frac{\partial^2 \tilde{u}_{1y}}{\partial t_o^2} &= -\frac{\partial^2 \tilde{d}}{\partial y_o \partial t_o} + \eta\frac{\partial}{\partial y_o}\left(\frac{\tilde{u}_{1z}}{c_{op}}\right)\\
\frac{1}{c_o}\frac{\partial^2 \tilde{u}_{1z}}{\partial t_o^2} &= -\frac{\partial^2 \tilde{d}}{\partial y_o \partial t_o} + \eta\frac{\partial}{\partial z_o}\left(\frac{\tilde{u}_{1z}}{c_{op}}\right) + \eta\frac{c_{op}}{c_o}\frac{\partial \tilde{d}}{\partial t_o}
\end{aligned} \tag{23}$$

where

$$\eta = (gl/c_o c_{op})[(2-\gamma)/2] + (1/2c_{op})(dc_o/dz_o) \tag{24}$$

This system of differential equations is in formal accordance with the corresponding results of Schrödinger[17] and of Stuff.[2] Now the function

$$\phi = \frac{1}{4\pi} f\left(t_o - \int_0^s \frac{c_{op}}{c_o(s)}\,ds\right)\bigg/\left(\frac{c_o}{c_{op}}\right)R \tag{25}$$

is introduced, which is analogous to the retarded potential in a homogeneous atmosphere. R is the instantaneous radius of the wavefront (Fig. 3). We have

$$R = (x_o^2 + y_o^2 + z_o^2)^{1/2}\frac{c_{op}}{c_o}\left(\frac{c_o}{c_{op}} + \tfrac{1}{4}A^2(x_o^2 + y_o^2 + z_o^2)\right)^{1/2} \tag{26}$$

Because of the linear sound velocity gradient [Eq. (3)], the wave fronts form spheres, which are not concentric.[16] s_x, s_y and s_z are the direction cosines of the wave front

$$s_x^2 + s_y^2 + s_z^2 = 1 \tag{27}$$

with

$$\begin{aligned}
s_x &= x_o/R\\
s_y &= y_o/R\\
s_z &= (z_o - \tfrac{1}{2}A\, c_{op}/c_o(x_o^2 + y_o^2 + z_o^2)/R
\end{aligned} \tag{28}$$

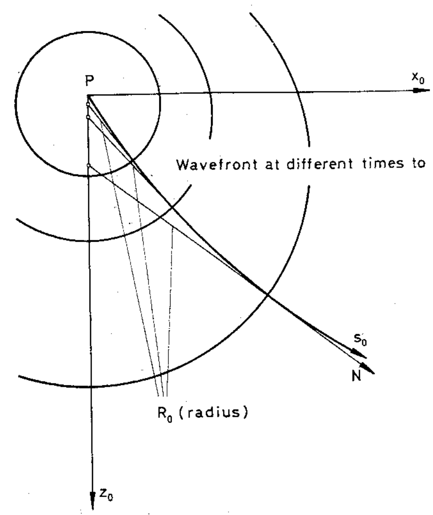

Fig. 3 The wavefronts of a spherical source in a polytropic atmosphere are not concentric spheres.

In case of a spherical source (Fig. 3), the following solution of the system of differential equations (22–23) applies

$$\begin{aligned}
\partial\tilde{d}/\partial t_o &= -(\partial^2\phi/\partial t_o^2)\\
\tilde{u}_{1x}/c_{op} &= (c_o/c_{op})(\partial\phi/\partial x_o) + (\eta s_x s_z \phi)\\
\tilde{u}_{1y}/c_{op} &= (c_o/c_{op})(\partial\phi/\partial y_o) + (\eta s_y s_z \phi)\\
\tilde{u}_{1z}/c_{op} &= (c_o/c_{op})(\partial\phi/\partial z_o) - [\eta(1 - s_z^2)\phi]
\end{aligned} \tag{29}$$

Here, terms of the order η^2, and those of the order of R^{-2} from terms multiplied by η, have been neglected. The latter is admissable, since the stratification terms multiplied by η are only important at large distances from the origin. In case of a continuous distribution of sources and sinks, the function ϕ has the form

$$\phi = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{f(\xi, t_o - \bar{s}(\xi, x_o, y_o, z_o))}{(c_o/c_{op})R(\xi, x_o, y_o, z_o)}\,d\xi \tag{30}$$

There are no stratification terms in ϕ and in the absolute value of the velocity vector $\tilde{u}_1$

$$|\tilde{u}_1| = (\tilde{u}_{1x}^2 + \tilde{u}_{1y}^2 + \tilde{u}_{1z}^2)^{1/2} \tag{31}$$

All terms of the order η on the right-hand side of Eq. (31) cancel each other. However, the direction of the velocity vector $\tilde{u}_1$ does depend on the stratification parameter η. The directions of $\tilde{u}_1$ and the wave-front normal are not always the same.[17] However, this effect is small compared with the first-order perturbations. Therefore, the stratification enters the problem under consideration mainly through the Eqs. (20). The function ϕ corresponds to the retarded potential of a homogeneous atmosphere.

Shock Waves

The formulas for the head wave can be easily derived or directly taken from Refs. 2 and 4. The first-order perturbation velocities of these papers have to be replaced by the corresponding velocities of the preceding section [Eqs. (20, 29 and 31)]. Weak shock waves are located at values for μ and ν satisfying the condition

$$\nu/\mu \ll 1 \tag{32}$$

(see also Fig. 2). Under this condition and with the aid of the bicharacteristics (Eq. 12), one obtains the first-order velocity perturbation from Eqs. (20) and (29), which is located in the direction of the acoustic perturbation front normal

$$u_{1N}/c_o(z_o) = [2/(\gamma-1)][c_1/c_o(z_o)] = \left[\frac{c_{op}p_{op}}{c_o p_o}\right]^{1/2} \frac{\partial\phi}{\partial t_o} \quad (33)$$

The differential equation for the head wave is taken from Oswatitsch[4]

$$\frac{d\nu}{d\mu} = \frac{\gamma+1}{8} \frac{[u_{1N}/c_o(z_o)]}{1+\dfrac{\partial t_1}{\partial \nu}} \quad (34)$$

This formula indicates that the shock slope is given by the bisector of the characteristic slopes in front of and behind the shock. In second order, the bisector rule is no longer valid, as shown by Van de Vooren and Dijkstra.[19] With the aid of Eqs. (14, 31, 32, and 34), the perturbation coordinates can be obtained by an integration over the bicharacteristic μ

$$s_1 = -t_1 = \frac{\gamma+1}{4}\int_{\mu_0}^{\mu} u_{1N}(\bar\mu,\nu)d\bar\mu + K_1(\nu) \quad (35)$$

The perturbation pressure p_1 divided by local static pressure $p_o(z_o)$ of the polytropic atmosphere can be obtained from Eqs. (21, 31, 32, and 33)

$$p_1/p_o(z_o) = \gamma(u_{1N}/c_0(z_o)) \quad (36)$$

Under the condition (32), the first-order position of the shock wave is represented by the acoustic perturbation front [Eq. (12)]

$$\mu = \bar s_o = \bar t_o \quad (37)$$

Let ξ_a describe the path of the body tip. If we use the body-fixed coordinate $\xi - \xi_a$ as variable of integration in Eq. (30), the potential ϕ, under the condition (32), can be separated into two functions. Each of these functions depends on one bicharacteristic only. Now replacing the derivatives with respect to t_o in Eq. (33) by those with respect to ν and introducing the separated functions, the following equation results for u_{1N}:

$$u_{1N}/c_o(z_o) = M^2(M/2)^{1/2}F(2\nu M)G'(\mu) \quad (38)$$

with

$$F(2\nu M) = \frac{1}{2\pi}\int_0^{2\nu M} \frac{s''(\xi-\xi_a)}{(2\nu M-(\xi_a-\xi))^{1/2}}d(\xi_a-\xi) \quad (39)$$

$G'(\mu)$ is determined in the following sections for some typical aircraft maneuvers.

Defining

$$G(\mu) = \int_0^{\mu} G'(\bar\mu)d\bar\mu \quad (40)$$

the connection between μ and ν on the shock wave is given by the solution of the shock-wave [Eq. (34)]

$$G(\mu) = \int_o^{\nu} F(2\bar\nu M)d\bar\nu/\{[(\gamma+1)/8]M^2(M/2)^{1/2}F^2(2\nu M)\} \quad (41)$$

which is similar to Whitham's result for the homogeneous atmosphere.[9]

In order to find an explicit formula, Whitham substituted the neutral Mach line ν_n for the upper limit ν in the integral of Eq. (41). It should be noted that the F function is zero on the neutral Mach line ν_n. The asymptotic formula of Whitham, therefore, predicts too large values for the pressure jump across the shock wave for distances of practical interest.[20] The asymptotic formula of Whitham can be easily improved by taking into account the distance function $G(\mu)$ for the characteristic ν^*, which hits the shock at a distance μ. The Whitham assumption is used as a first step for calculating ν^*

$$F(2\nu^* M) = \left(\frac{\int_0^{\nu_n} F(2\bar\nu M)d\bar\nu}{\dfrac{(\gamma+1)}{8}M^2\left(\dfrac{M}{2}\right)^{1/2}G(\mu)}\right)^{1/2} \quad (42)$$

The values for the bow shock calculated in this way are sufficiently accurate for distance of about 20 body lengths or larger (see Fig. 5). If the flow behind the rear shock wave is undisturbed, the formulas for the head wave apply for the rear shock wave, too.[2]

However, this method based on the above-mentioned assumption yields wrong results near the body, whereas the more complicated method proposed by Whitham[9] for the calculation of the rear shock wave can be used in the near and the far field. For example, an axisymmetric parabolic arc is taken, with the F function

$$F(2\nu M) = 2tg^2\delta(2\nu M)^{1/2}[1 - 8\nu M + 64/5(\nu^2M^2)] \quad (43)$$

where δ is the apex half angle. The distance function $G'(\mu)$ [Eq. (38)] depends on both the temperature stratification and the flight maneuvers; therefore, it has to be determined for every detailed maneuver separately.

Shock Waves in Case of Accelerated or Decelerated Flight along a Straight Line

The body is flying with supersonic speed accelerated or decelerated along a straight line (see Fig. 4). The flight path is located in a vertical plane denoted by x_o and z_o, and is inclined by the angle Φ against the horizontal x_o axis. The perturbations propagating along the ray s will be considered. The vertical plane containing the ray makes a horizontal angle θ with the x_o axis.

Flight at Constant Altitude ($\Phi = 0$)

Shock waves propagating below the flight path are indicated by $\theta = 0$. The retarded potential at a point $P(x_o, z_o, t_o)$ in the vertical plane is obtained by an integral over the distribution of singularities on the flightpath at position ξ and time τ. The equation of the Mach conoid that is placed with its tip on such a singularity is

$$\xi - x_o = -\left(\frac{2}{A^2}\frac{c_o}{c_{op}}\{\cosh|A(t_o-\tau)| - 1\} - z_o^2\right)^{1/2} \quad (44)$$

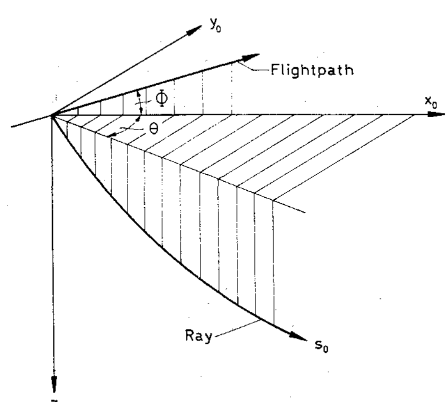

Fig. 4 The flight path is located in the horizontal x_o, z_o-plane and may be inclined by the angle Φ against the x_o-axis.

In presence of a constant acceleration B_t in the tangential direction the coordinate $\xi - \xi_a$ is given by

$$\xi - \xi_a = \xi - x_p - M(\tau - t_p) - (B_t/2)(\tau - t_p)^2 \quad (45)$$

Under the condition (32), Eq. (44) can be expanded in a series:

$$\xi - x_p = -2\nu M + M(\tau - t_p) + (\tau - t_p)^2 \frac{AM}{2tgh|A\mu|}(\beta^2 - M\beta \tanh|A\mu|) + \cdots \quad (46)$$

with $\beta = (M^2 - 1)^{1/2}$.

Then the coordinate $\xi - \xi_a$ can be expressed

$$\xi - \xi_a = -2\nu M + (\tau - t_p)^2 \frac{AM}{2tgh|A\mu|}\left(\beta^2 - \left\{M\beta A + \frac{B_t}{M}\right\}\frac{1}{A}\tanh|A\mu|\right) + \cdots \quad (47)$$

Replacing the integration variable ξ in Eq. (30) by $\xi - \xi_a$, the distance function can be written with the aid of Eqs. (33) and (38) as follows:

$$G'(\mu) = \frac{c_{op}}{c_o}\left(\frac{c_{op}p_{op}}{c_o p_o}\right)^{1/2} \times \left[L(\mu)\left(\beta^2 - \left\{M\beta A + \frac{B_t}{M}\right\}\frac{1}{A}\tanh|A\mu|\right)\right]^{-1/2}$$

with

$$L(\mu) = \frac{1}{A}\sinh|A\mu|\cosh|A\mu|. \quad (48)$$

The denominator of Eq. (48) is zero if the waves form a cusp. However, this singularity is an integrable one, and the shock waves behind the cusp can be calculated.

In the case $A = 0$, the formulas for the isothermal atmosphere are obtained,[2] and in the case $A = 0$ and $\alpha = 0$, the formulas for the homogeneous atmosphere are reproduced.[9] Now the bicharacteristic ν, which hits the shock at a distance given by μ, can be obtained by an iteration of Eq. (41) with the aid of Eq. (48). Then the perturbation pressure and coordinates are determined by (35), (36) and (38). The sonic-boom signatures of a parabolic arc are shown in Fig. 5–6.

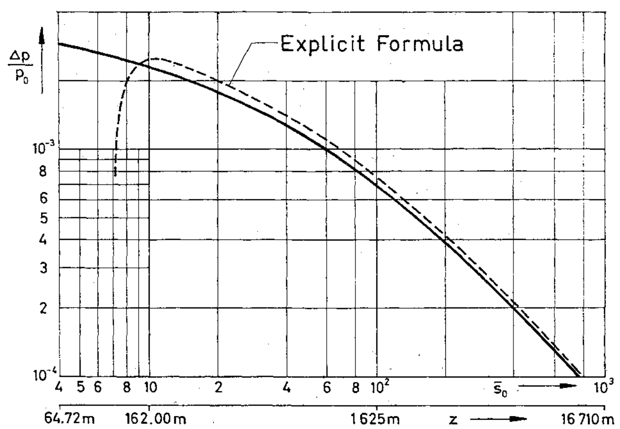

Fig. 5 **The pressure jump Δp across the head wave below the flight path ($\theta = 0$) divided by the local static pressure p_o is plotted against ray distance $\bar{s}_o$ and altitude z for steady flight at $M = 1{,}7$. The length of the parabolic arc is $l = 20\,m$ and its apex half angle $\delta = 5°$. The polytropic stratification is defined by a sound velocity of $c_{op} = 300$ m/sec at flight altitude, a sound velocity gradient of $m = 4.10^{-3}$ 1/sec and the gravity acceleration.**

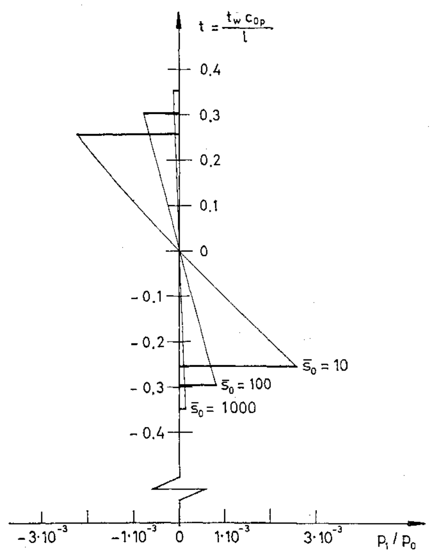

Fig. 6 ***N*-wave at different distances measured along the ray.**

Shock-wave propagation in lateral direction ($\theta \neq 0$) means that the vertical plane containing the ray (Fig. 4) does not contain the flight path. Therefore, the exact potential at a point $P(x_o, y_o, z_o, t_o)$ can be obtained rigorously only by a four-dimensional treatment of the integration over the distribution of sources and sinks on the flight path. However, the distance function must approximate the corresponding formula for the homogeneous atmosphere in the near field. On the other hand, $G'(\mu)$ must have a singularity, if, according to geometrical acoustics, a cusp occurs. The considerations just described lead to

$$G'(\mu) = \frac{c_{op}}{c_o}\frac{\bar{\beta}}{\beta}\left(\frac{c_{op}p_{op}}{c_o p_o}\right)^{1/2} \left[L(\mu)\left(\bar{\beta}^2 - \left\{M\cos\theta\bar{\beta}A + \frac{B_t}{M}\right\}\frac{1}{A}\tanh|A\mu|\right)\right]^{-1/2}$$

with

$$\bar{\beta} = (M^2\cos^2\theta - 1)^{1/2} \quad (49)$$

β and $L(\mu)$ are defined in Eqs. (46) and (48) respectively.

Flight Path Inclined ($\Phi \neq 0$)

If the flight path of the body is inclined (see Fig. 4) against horizontal x_o axis, it should be noted that c_{op} is a function of t_p.[16] The quantity ξ_p is introduced

$$\xi_p = t_p + (B_t/2)(t_p^2),\ x_p = \xi_p\cos\Phi,\ z_p = \xi_p\sin\Phi \quad (50)$$

(see Eq. (6)).

For $\theta \neq 0$, the distance function can be determined similar to Eq. (49).

$$G'\mu) = \frac{c_{op}}{c_o}\left(\frac{c_{op}p_{op}}{c_o p_o}\right)^{1/2}\frac{\bar{\beta}}{\beta}\left[L(\mu)\left(\bar{\beta}^2 - \left\{(M\cos\theta\cos\Phi\bar{\beta} + M\sin\Phi)A + \frac{B_t}{A}\right\}\frac{1}{A}\tanh|A\mu|\right)\right]^{-1/2}$$

with

$$\bar{\beta} = (M^2(\cos^2\Phi\cos^2\theta + \sin^2\Phi) - 1)^{1/2} \quad (51)$$

Curved Flight Path

In this case, the sonic boom position can be taken again from geometrical acoustics.[16] It should be noted that y_p is a function of t_p, since there is a radial acceleration in the direction normal to the flight path

$$d^2y_p/dt_p^2 = B_r = M^2/R_K \tag{52}$$

For a curved flight with the flight path inclined, $(\Phi \neq 0)$ $G'(\mu)$ is then given by

$$G'(\mu) = (c_{op}/c_o)(c_{op}p_{op}/c_o p_o)^{1/2}(\bar{\beta}/\beta)$$

$$\cdot\left[L(\mu)\left(\bar{\beta}^2 - \left\{(M\cos\theta\cos\Phi\bar{\beta} + M\sin\Phi)A + \frac{1}{M}(B_t + B_r\tan\theta)\right\}\frac{1}{A}\tanh|A\mu|\right)\right]^{-1/2}$$

$$\text{with } \bar{\beta} = (M^2(\cos^2\Phi\cos^2\theta + \sin^2\Phi) - 1]^{1/2} \tag{53}$$

β and $L(\mu)$ are defined in Eqs. (46) and (48), respectively. Note that there is no influence of the curvature on the sonic boom propagating in the tangential plane of the flight path since $\theta = 0$.

Conclusions

Solutions for singularities in a polytropic atmosphere are derived. Using the analytic methods of singularities and of characteristics,[4,5] the sonic boom of the body is obtained analytically. A parabolic arc is chosen as an example. The asymptotic Whitham formula for the bow wave is improved by an explicit formula giving sufficiently accurate results for distances of about 20 body lengths or larger.

The shock-wave propagation below the flight path is derived rigorously. But to avoid a four-dimensional treatment, the shock-wave propagation into the lateral direction is concluded from that it must have a singularity if the waves form a cusp, and must approximate the propagation of a homogeneous atmosphere in the near field. The method presented can be extended to describe the influence of a three-dimensional configuration, if the configuration is replaced by the corresponding distribution of singularities. The sonic boom resulting from lift can be investigated, making use of ideas of Oswatitsch and Sun. [21,22] The method presented can also be generalized to a second-order sonic-boom theory with the aid of ideas of Schneider[10] and of Landhal, Ryhming and Lofgren.[11]

There are several other applications of the analytic first-order solution presented. It is valid beyond the scope of geometric acoustics, since it satisfies the wave equation, and can also be applied as initial solution in the theory of propagation, scattering, and absorption of noise in a polytropic atmosphere.

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